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GUARANTEED RESULT IN A DIFFERENTIAL GAME OF GROUP PURSUIT WITH TERMINAL PAYOFF FUNCTION†

I. S. RAPPOPORT and A. A. CHIKRII

Kiev

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A method is proposed for solving a differential game of group pursuit with a terminal payoff function, consisting of the systematic utilization of the idea of Fenchel–Moreau duality [1] applied to the general scheme of the resolving function method [2]. It is shown that if the parameters of a quasi-linear control process and terminal payoff function satisfy certain sufficient conditions, the group pursuit will come to a successful end. The guaranteed termination time is determined on the basis of information about the initial state of the process and the prehistory of the evader's control. The results are illustrated for a model example.

In [3] we proposed a method of solving a differential two-person pursuit game with a terminal payoff function (PF). The basic idea of the method is that a resolving function can be expressed in terms of the conjugate to the PF and, using the involutive property of the conjugation operator for a convex closed function, one obtains a guaranteed estimate of the terminal value of the PF. However, attempts to employ the method to solve a differential group-pursuit game with a terminal PF encounter fundamental difficulties, related to the fact that the PF in this case is defined as the minimum of a finite number of convex functions and need not itself be convex.

These difficulties are overcome by the method proposed in this paper.

The paper is an extension of the idea in [2–5]; it is closely related to the research reported in [6–11] and indicates new possibilities for applying convex analysis to the solution of many-person game problems of control.

1. FORMULATION OF THE PROBLEM AND AUXILIARY RESULTS

Consider a conflict-controllable process described by a system of quasi-linear differential equations

$$\dot{z}_i = A_i z_i + \varphi_i(u_i, v), \quad z_i \in R^{n_i}, \quad u_i \in U_i, \quad v \in V \tag{1.1}$$

where A_i is a square matrix of order n_i, φ_i: $U_i \times V \to R^{n_i}$ is a function continuous jointly in all its variables, and U_i and V are non-empty compact sets in the Euclidean space R^{n_i}. Here and henceforth, $i = 1, \ldots, \nu$.

We are given a payoff function $\sigma(z)$, which determines the time at which the game ends and may be represented in the form

$$\sigma(z) = \min_{1 \leqslant i \leqslant \nu} \sigma_i(z_i), \quad z = (z_1, \ldots, z_\nu), \quad z_i \in R^{n_i}, \quad \sigma_i: R^{n_i} \to R^1 \tag{1.2}$$

If $z_i(t) = z_i(z_i^0, u_i^t(\cdot), v^t(\cdot))$ is a trajectory of system (1.1) corresponding to an initial state z_i^0 and the players' chosen controls $u_i^t(\cdot) = \{u_i(\tau): u_i(\tau) \in U_i, \tau \in [0, t]\}$; $v^t(\cdot) = \{v(\tau); v(\tau) \in V, i \in [0, t]\}$, the game will be assumed to have ended at time T if

$$\sigma(z(T)) \leqslant 0, \quad z(T) = (z_1(T), \ldots, z_\nu(T)) \tag{1.3}$$

The object of the group of pursuers $(u_1, \ldots, u_\nu)$ is to make the game end, while that of the evader (v) is to achieve the opposite.

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We shall assume that the controls used during the game by the pursuer and the evader are Lebesgue-measurable functions of time. We will take the side of the pursuer and indicate sufficient conditions which, if satisfied by the parameters of the process (1.1) and of the terminal payoff function (1.2), will ensure that the game (1.1), (1.3) comes to an end. At the same time we will determine the guaranteed termination time on the basis of information about the initial state $z^0 = (z_1^0, \ldots, z_\nu^0)$ and the prehistory of the evader's control $v(\cdot)$.

We shall assume that the functions $\sigma_i(z_i)$ are convex and Lipschitz continuous

$$|\sigma_i(z_i) - \sigma_i(x_i)| \leqslant l_i \|z_i - x_i\|, \quad l_i \geqslant 0, \quad z_i, x_i \in R^{n_i} \tag{1.4}$$

It is well known from convex analysis [1] that these functions may be expressed in the form

$$\sigma_i(z_i) = \max_{p_i \in \operatorname{dom} \sigma_i^*} [(p_i z_i) - \sigma_i^*(p_i)] \tag{1.5}$$

where $\sigma_i^*(p_i)$, $\sigma_i^*: R^{n_i} \to R^1$ is the function conjugate to $\sigma_i(z_i)$ defined by the equality

$$\sigma_i^*(p_i) = \sup_{z_i \in R^{n_i}} [(p_i, z_i) - \sigma_i(z_i)], \quad p_i \in R^{n_i} \tag{1.6}$$

and dom σ_i^* is the effective set [1] of the function $\sigma_i^*(p_i)$, that is

$$\operatorname{dom} \sigma_i^* = \{p_i \in R^{n_i}: \sigma_i^*(p_i) < +\infty\}$$

Under these conditions, it follows from (1.4) that dom σ_i^* is a compact set [1].

We require in addition that the functions $\sigma_i(z_i)$ should be bounded below. Then, by (1.6), we have

$$-\sigma_i^*(0) = \inf_{z_i \in R^{n_i}} \sigma_i(z_i)$$

and so the set dom σ_i^* contains zero.

Let L_i be the linear span of the set dom σ_i^* [1] and let π_i be the orthogonal projection operator from R^{n_i} onto the subset L_i. Using representation (1.5), one can verify the relation

$$\sigma_i(z_i) = \sigma_i(\pi_i z_i), \quad z_i \in R^{n_i} \tag{1.7}$$

2. SCHEME OF THE METHOD AND MAIN RESULT

We introduce multivalued mappings

$$W_i(t, v) = \bigcup_{u_i \in U_i} W_i(t, u_i, v), \quad W_i(t) = \bigcap_{v \in V} W_i(t, v)$$

where $W_i(t, u_i, v) = \pi_i \Phi_i(t)\, \varphi_i(u_i, v)$; $\Phi_i(t) = \exp(tA_i)$, $t \geqslant 0$.

It is assumed that the parameters of process (1.1) satisfy Pontryagin's condition [2, 6, 8], which means that $W_i(t) \neq 0$ for all $t \geqslant 0$.

Since the mappings $W_i(t)$ are upper semicontinuous, each of them contains at least one Borel selector [2, 9, 12]. Denoting the sets of these selectors by Γ_i, we fix one selector in each $\gamma_i(\cdot) \in \Gamma_i$ and define

$$\xi_i(t, z_i, \gamma(\cdot)) = \pi_i \Phi_i(t)\, z_i + \int_0^t \gamma(\tau)\, d\tau$$

We define a resolving function for each by

$$\beta(t, \tau, z_i, v, \gamma(\cdot)) = \tag{2.1}$$

$$= \sup\{\beta \geqslant 0: \min_{u_i \in U_i} \max_{p_i \in \operatorname{dom} \sigma_i^*} [(p_i, I_i) + \beta_i J_i)] \leqslant 0\}, \quad t \geqslant \tau \geqslant 0, \quad z_i \in R^{n_i}, \ v \in V$$

$$I_i = W_i(t - \tau, u_i, v) - \gamma_i(t - \tau), \quad J_i = (p_i, \xi_i(t, z_i, \gamma_i(\cdot)) - \sigma_i^*(p_i))$$

It follows directly from Pontryagin's condition that

$$\min_{u_i \in U_i} \max_{p_i \in \operatorname{dom} \sigma_i^*} (p_i, l_i) \leq 0, \quad \forall t \geq \tau \geq 0, \quad v \in V \tag{2.2}$$

Consequently, if Pontryagin's condition is satisfied, the inequalities in (2.1) are true for at least the zero values of the resolving functions. We also note that if $\sigma_i(\xi_i(t, z_i, \gamma_i(\cdot)) \leq 0$, then $\beta_i(t, \tau, z_i, v, \gamma(\cdot)) = +\infty$ for all $v \in V$, $t \in [0, t]$. But if it is true that for some t, $z_i \in R^{n_i}$, $\gamma(\cdot) \in \Gamma_i$, $\sigma_i(\xi_i(t, z_i, \gamma_i(\cdot))) > 0$, the resolving functions $\beta_i(t, \tau, z_i, v, \gamma_i(\cdot))$ take finite values and are bounded uniformly with respect to $t \in [0, t]$, $v \in V$.

Lemma 2.1. Suppose that the parameters of process (1.1) satisfy Pontryagin's condition, and that the payoff function (1.2) satisfies the conditions of Section 1. Then, if it is true that for some $t > 0$, $z_i \in R^{n_i}$ and $\gamma_i(\cdot) \in \Gamma_i$, $\sigma(\xi_i(t, z_i, \gamma_i(\cdot))) > 0$, then

$$\beta_i(\tau, v) = \beta_i(t, \tau, z_i, v, \gamma_i(\cdot)) \tag{2.3}$$

is a Borel function jointly with respect to all its variables (τ, v) on the set $[0, t] \times V$.

Proof. Fix $t > 0$, $z_i \in R^{n_i}$ and $\gamma_i(\cdot) \in \Gamma_i$ such that $\sigma_i(\xi_i(t, z_i, \gamma_i(\cdot))) > 0$. It follows from our assumptions concerning the parameters of process (1.1) and the payoff function (1.2) that the functions $\psi_i(u_i, v, p_i, \tau, \gamma_i, \beta_i) = (p_i, W_i(t - \tau, u_i, v) - \gamma_i) + \beta_i J_i$ are continuous jointly in all variables, and consequently [10] the same is true of the function

$$\Psi_i(v, \tau, \gamma_i, \beta_i) = \min_{u_i \in U_i} \max_{p_i \in \operatorname{dom} \sigma_i^*} \psi_i(u_i, v, p_i, \tau, \gamma_i, \beta_i)$$

Then, as shown in [9], the multivalued mappings

$$B_i(\tau, v, \gamma_i) = \{\beta_i \geq 0 : \Psi_i(v, \tau, \gamma_i, \beta_i) \leq 0\}$$

are upper semicontinuous, and their selectors

$$\beta_i(\tau, v, \gamma_i) = \sup\{\beta_i : \beta_i \in B_i(\tau, v, \gamma_i)\}$$

are Borel functions jointly with respect to all their variables.

Therefore, by the superposition property for Borel functions [12], applied to the functions (2.3)

$$\beta_i(\tau, v) = \beta_i(\tau, v, \gamma_i(t - \tau)) = \beta_i(t, \tau, z_i, v, \gamma_i(\cdot))$$

are Borel functions on the set $[0, t] \times V$.

Consider the function

$$T(z, \gamma(\cdot)) = \inf\{t \geq 0 : \inf_{v(\cdot) \in \Omega_V} \max_{1 \leq i \leq \nu} H_i(t) \geq 1\} \tag{2.4}$$

$$H_i(t) = \int_0^t \beta_i(t, \tau, z_i, v(\cdot), \gamma_i(\cdot))\, d\tau$$

$$z = (z_1, \ldots, z_\nu), \quad \gamma(\cdot) = (\gamma_1(\cdot), \ldots, \gamma_\nu(\cdot)), \quad z_i \in R^{n_i}, \quad \gamma_i(\cdot) \in \Gamma_i$$

where Ω_V is the set of all measurable functions whose values lie in V.

If the inequality in the braces in (2.4) is not true for all $t \geq 0$, we put $T(z, \gamma(\cdot)) = +\infty$.

If i exists such that $\beta_i(t, \tau, z_i, v, \gamma_i(\cdot)) = +\infty$ for $\tau \in [0, t]$, $v \in V$, we agree to put $H_i(t) = +\infty$ and the inequality in (2.4) will hold automatically.

Theorem 2.1. Suppose that the parameters of process (1.1) satisfy Pontryagin's condition, the payoff function (1.2) satisfies the conditions of Section 1, and for some $z^0 = (z_1^0, \ldots, z_\nu^0)$, $\gamma^0(\cdot) = (\gamma_1^0, \ldots, \gamma_\nu^0)$, $z_1^0 \in R^{n_i}$, $\gamma_i^0(\cdot) \in \Gamma_i$ we have the inequality $T(z^0, \gamma^0(\cdot)) < +\infty$. Then the game may be completed at time $T(z^0, \gamma^0(\cdot))$ from the initial state z^0.

Proof. Set $T = T(z^0, \gamma^0(\cdot))$. Let $v(\tau)$, $v(\tau) \in V$, $\tau \in [0, T]$ be an arbitrary measurable function. We will demonstrate the pursuers' choice of controls.

Let us consider the case

$$\sigma(\xi(T,z^0,\gamma^0(\cdot)))>0,\quad \xi(T,z^0,\gamma^0(\cdot))=(\xi_1(T,z_1^0,\gamma_1^0(\cdot)),\dots,\xi_\nu(T,z_\nu^0,\gamma_\nu^0(\cdot)))$$

Define a control function by

$$h(t)=1-\max_{1\le i\le \nu} H_i^0(t)$$

$$H_i^0(t)=\int_0^t \beta_i(T,\tau,z_i^0,v(\tau),\gamma_i^0(\cdot))\,d\tau$$

The function $h(t)$ is continuous and non-increasing, and $h(0)=1$. It follows from the definition of T that t_* exists, $0\le t_*\le T$, for which

$$h(t_*)=0 \tag{2.5}$$

Consider the multivalued mappings

$$U_i^1(\tau,v)=\{u_i\in U_i:\ \max_{p_i\in \mathrm{dom}\,\sigma_i^*}[(p_i,I_i^0)+\beta_i(T,\tau,z_i^0,v,\gamma_i^0(\cdot))\,J_i^0]\le 0\}$$

$$I_i^0=W_i(T-\tau,u_i,v)-\gamma_i^0(T-\tau),\quad J_i^0=(p_i,\xi_i(T,z_i^0,\gamma_i^0(\cdot)))-\sigma_i^*(p_i)$$

$$0\le\tau\le t_*,\quad v\in V$$

These mappings are Borel jointly in all their variables.

Indeed, the functions

$$\varkappa_i(u_i,v,\tau,\gamma_i,\beta_i)=\max_{p_i\in \mathrm{dom}\,\sigma_i^*}[(p_i,W_i(T-\tau,u_i,v)-\gamma_i)+\beta_i J_i^0]$$

are continuous jointly in their variables. Hence the mappings

$$U_i(\tau,v,\gamma_i,\beta_i)=\{u_i\in U_i:\varkappa_i(u_i,v,\tau,\gamma_i,\beta_i)\le 0\}$$

are upper semicontinuous [9] and therefore the multivalued mappings

$$U_i^1(\tau,v)=U_i(\tau,v,\gamma_i(T-\tau),\beta_i(T,\tau,z_i^0,v,\gamma_i^0(\cdot)))$$

are Borel jointly in the variables (τ,ν), as superpositions of semicontinuous and Borel mappings [12].

Then the selectors

$$u_i^1(\tau,v)=\mathrm{lex\,min}\,U_i^1(\tau,v),\quad 0\le\tau\le t_*,\quad v\in V$$

are Borel functions in (τ,ν) [2, 12].

Consider the multivalued mappings

$$U_i^2(\tau,v)=\{u_i\in U_i:I_i^0=0\},\quad t_*\le\tau\le T,\quad v\in V$$

These mappings are Borel jointly in (τ,ν) [2, 9]. Then the selectors

$$u_i^2(\tau,v)=\mathrm{lex\,min}\,U_i^2(\tau,v),\quad t_*\le\tau\le T,\quad v\in V$$

are Borel functions in (τ,ν) [2, 12].

The pursuers' controls in the interval $[0,T]$ are defined as

$$u_i(\tau)=\begin{cases}u_i^1(\tau,v(\tau)), & \tau\in[0,t_*]\\ u_i^2(\tau,v(\tau)), & \tau\in[t_*,T]\end{cases}$$

The functions $u_i(\tau)$ are measurable [2, 12].

We now consider the case when $\sigma(\xi(T, z^0, \gamma^0(\cdot))) \leqslant 0$. Then a number j ($j = 1, \dots, \nu$) exists such that $\sigma_j(\xi_j(T, z^0_j, \gamma^0_j(\cdot))) \leqslant 0$ and the jth pursuer's control in $[0, T]$ is defined as $u_j(\tau) = u^2_j(\tau, v(\tau))$. This function is also measurable [2, 12]. The controls of the remaining pursuers are arbitrary.

With the control of the pursuer group chosen in this way, relation (1.3) will hold at time T on the appropriate trajectories of process (1.1), whatever control is chosen by the evader.

Indeed, Cauchy's formula for process (1.1) implies the representation

$$\pi z_i(T) = \xi_i(T, z_i^0, \gamma_i^0(\cdot)) + \int_0^T I_i^0 d\tau \tag{2.6}$$

Let $\sigma(\xi(T, z^0, \gamma^0(\cdot))) \leqslant 0$. Then a number j ($j = 1, \dots, \nu$) exists such that $\sigma_j(\xi_j(T, z_j^0, \gamma_j^0(\cdot))) \leqslant 0$. In accordance with the control law for the pursuer group, we have $\pi z_j(T) = \xi_j(T, z_j^0, \gamma_j^0(\cdot))$. Hence the truth of (1.3) follows immediately from (1.7) and representation (1.2).

Now let $\sigma(\xi(T, z^0, \gamma^0(\cdot))) > 0$. Then by (1.2) we have $\sigma_j(\xi_i(T, z_i^0, \gamma_i^0(\cdot))) > 0$. It follows from (2.5) that a number j ($j = 1, \dots, \nu$) exists such that

$$1 - H_j^0(t_*) = 0 \tag{2.7}$$

Let us follow the jth pursuer. Relations (1.7) and (2.6) yield

$$\sigma_j(z_j(T)) = \max_{p_j \in \operatorname{dom} \sigma_j^*} \left[J_j^0 + \int_0^T (p_j, I_j^0)\, d\tau \right] \tag{2.8}$$

Adding to and subtracting from the bracketed expression the quantity $J_j H_j^0(t_*)$, we conclude that if the pursuer group choose their control in accordance with the above law, the jth pursuer can guarantee that at time T we have the inequality

$$\sigma_j(z_j(T)) \leqslant \sigma_j(\xi_j(T, z_j^0, \gamma_j^0(\cdot)))[1 - H_j^0(t_*)]$$

Consequently, taking (2.7) and (1.2) into account, we obtain (1.3).

Corollary 2.1. Suppose that the parameters of process (1.1) satisfy Pontryagin's condition, and that the payoff function (1.2) satisfies the assumptions of Section 1. Then, if the pursuer group employs the control laws described in the proof of Theorem 2.1, for any T, $0 < T < T(z^0, \gamma^0(\cdot))$ a number j ($j = 1, \dots, \nu$) exists such that

$$\sup_{v(\cdot) \in \Omega_V} \sigma(z(T)) \leqslant \sigma_j(\xi_j(T, z_j^0, \gamma_j^0(\cdot))) \left[1 - \inf_{v(\cdot) \in \Omega_V} \max_{1 \leqslant i \leqslant \nu} H_i^0(T) \right]$$

The proof is analogous to that of Theorem 2.1, provided the control function is defined as

$$h(t) = \inf_{v(\cdot) \in \Omega_V} \max_{1 \leqslant i \leqslant \nu} H_i^0(T) - \max_{1 \leqslant i \leqslant \nu} H_i^0(t)$$

3. GENERALIZED DISTANCE AND RESOLVING FUNCTIONS

Let M_i^* be convex closed sets and let S_i be convex closed bounded sets whose interiors contain zero. Then for all $z = (z_1, \dots, z_\nu)$, $z_i \in R^{n_i}$ one can define a generalized distance function [10]

$$d_S(z | M^*) = \min_{1 \leqslant i \leqslant \nu} d_S(z_i | M_i^*) \tag{3.1}$$

$$S = S_1 \times \ldots \times S_\nu, \quad M^* = \bigcup_{i=1}^{\nu} M_i^*, \quad d_{S_i}(z_i | M_i^*) = \inf\{\rho_i \geqslant 0 \colon\ z_i \in M_i^* + \rho_i S_i\}$$

It can be shown that the generalized distance function (3.1) satisfies the conditions of Section 1.

Let us calculate the functions conjugate to the functions $d_{S_i}(z_i | M_i^*)$.

We first note that

$$d_{S_i}(z_i | M_i^*) = \inf\{\mu_{S_i}(z_i - m): m \in M_i^*\}$$

where $\mu_{S_i}(x_i) = \inf\{\mu_i \geqslant 0: x_i \in \mu_i S_i\}$ is the calibration function of the set S_i [10].

Therefore, relying on the definition of the infimal convolution operation [1, 9], we have

$$d_{S_i}(z_i | M_i^*) = (f_i \Box g_i)(z_i) = \inf\{f_i(z_i - y_i) + g_i(y_i): y_i \in R^{n_i}\}$$

where $\Box$ is the symbol of the infimal convolution operation [1], $f_i(x_i) = \mu_{S_i}(x_i)$ is the calibration function of the set S_i, $x_i \in R^{n_i}$ and $g_i(y_i) = \delta(y_i | M_i^*)$ is the indicator function of the set M_i^* [1].

By the duality theorem for the operations of addition and infimal convolution [1], we obtain the following formula for the conjugate functions

$$d_{S_i}^*(\cdot | M_i^*)(p_i) = f_i^*(p_i) + g_i^*(p_i) = \begin{cases} C(M_i^*, p_i), & p_i \in S_i^0 \\ +\infty, & p_i \notin S_i^0 \end{cases}$$

where $S_i^0 = \{p_i \in R^{n_i}: (p_i, x_i) \leqslant 1 \text{ for each } x_i \in S_i\}$ is the polar of the set S_i [1], $f_i^*(p_i) = \delta(p_i | S_i^0)$ is the indicator function of the polar of the set S_i, and $g_i^*(p_i) = C(M_i^*, p_i)$ is the support function of the set M_i^*.

Here we have used the fact that the calibration function of S_i is the support function of the polar S_i^0 [1], as well as the duality of the indicator and support functions of a convex closed set [1].

Thus, taking (1.5) into consideration, we have

$$\operatorname{dom} d_{S_i}^*(\cdot | M_i^*)(p_i) = S_i^0, \quad d_{S_i}(z_i | M_i^*) = \max_{p_i \in S_i^0}[(p_i, z_i) - C(M_i^*, p_i)]$$

Using this representation one can prove the following.

Lemma 3.1. Let X_i be a compact set, M_i^* be a convex closed set and S_i be a convex compact set with zero in its interior. Then a necessary and sufficient condition for $X_i \cap M_i^* \neq 0$ is that

$$\min_{z_i \in X_i} \max_{p_i \in S_i^0}[(p_i, z_i) - C(M_i^*, p_i)] \leqslant 0$$

where S_i^0 is the polar of the set S_i.

Let us take M_i^* to be cylindrical sets of the form $M_i^* = M_i^0 + M_i$, where M_i^0 is a linear subspace of R^n, M_i is a convex compact subset of the orthogonal complement L_i of M_i^0 in R^{n_i}. Relation (2.1) then implies an expression for the resolving functions $\beta_i(t, \tau, z_i, v, \gamma_i(\cdot))$

$$\sup\{\beta_i \geqslant 0: \min_{u_i \in U_i} \max_{p_i \in S_i^0 \cap L_i} [(p_i, W_i(t-\tau, u_i, v) - \gamma_i(t-\tau)) +$$

$$+ \beta_i[(p_i, \xi_i(t, z_i, \gamma(\cdot))) - C(M_i, p_i)]] \leqslant 0\}, \quad t \geqslant \tau \geqslant 0, \quad z_i \in R^{n_i}, \quad v \in V$$

where S_i is a convex subset of R^{n_i} whose interior contains the zero of the space.

Using Lemma 3.1, it can be shown that these functions are identical with the resolving functions $\alpha_i(\tau, z_i, v, \gamma_i(\cdot))$ defined by the formula [2]

$$\sup\{\alpha_i \geqslant 0: [W_i(t-\tau, v) - \gamma_i(t-\tau)] \cap \alpha[M_i - \xi_i(t, z_i, \gamma_i(\cdot))] \neq 0\}$$

We have thus established the relation between our results and the general scheme of the method of resolving functions for group pursuit [2, 5].

The determination of resolving functions may be a far from simple task. Under some special conditions on the parameters of process (1.1) and the payoff function (1.2) the problem may be simplified.

Lemma 3.2. Let process (1.1) be linear, that is, the functions $\varphi_i: U_i \times V \to R^{n_i}$ have the form $\varphi_i(u_i, v) = u_i - v$, and assume that Pontryagin's condition is satisfied. The terminal set M^* is the union of sets M_i^*, $M_i^* \subset R^{n_i}$, each of which may be expressed as $M_i^* = M_i^0 + l_i S_i$, where M_i^0 is a linear subspace of R^{n_i}, l_i is a non-negative number, and S_i is a convex compact subset of the orthogonal complement L_i of M_i^0

in R^{n_i}, whose interior contains the zero of the subspace L_i. Continuous positive functions $r_i(t)$ exist such that $\pi_i\Phi_i(t)U_i = r_i(t)S_i$. Then the resolving functions

$$\beta_i(t,\tau,z_i,v,\gamma_i(\cdot)), t \geq \tau \geq 0,\quad z_i \in R^{n_i},\quad v \in V,\quad \gamma_i(\cdot)\in\Gamma_i$$

for $\mu_{S_i}(\xi_i(t, z_i, \gamma_i(\cdot))) > l_i$ are the largest positive roots of the equation

$$\mu_{S_i}(\chi_i(\beta_i)) = r_i(t-\tau)+\beta_i l_i$$

considered as an equation in β_i, where $\chi_i(\beta_i) = \pi_i\Phi_i(t-\tau)v + \gamma_i(t-\tau) - \beta_i\xi_i(t, z_i, \gamma_i(\cdot))$.

Proof. Relying on the assumptions of the lemma, we conclude from (2.1) that the resolving functions $\beta_i(t, \tau, z_i, v, \gamma_i(\cdot))$ with their argument values fixed are the maximum numbers β_i such that

$$\min_{u_i\in S_i}\max_{p_i\in S_i^0}[(p_i, r_i(t-\tau)u_i - \pi_i\Phi_i(t-\tau)\, v - \gamma_i(t-\tau)) + \beta_i[(p_i, \xi_i(t,z_i,\gamma_i(\cdot))) - C(l_i S_i, p_i)]] \leq 0$$

These relations are equivalent to the following ones

$$\max_{p_i\in S^0}[(-p,\chi_i(\beta_i)) - (r_i(t-\tau)+\beta_i l_i)\mu_{S_i^0}(p_i)] \leq 0 \tag{3.2}$$

Here we have used the Minimax theorem [1, 10] and the following well-known relation from convex analysis [1]

$$C(S_i, p_i) = \mu_{S_i^0}(p_i),\quad p_i \in L_i \tag{3.3}$$

Using this relation and inequality (3.2), we obtain the required result.

Corollary 3.1. Under the assumptions of Lemma 3.2, let S_i be the unit sphere of the space L_i with centre at zero. Then the resolving functions are the largest positive roots of the quadratic equations

$$\|\chi_i(\beta_i)\| = r_i(t-\tau)+\beta_i l_i$$

considered as equations in β_i.

The proof follows from (3.3), taking into account the fact that the support function of the unit sphere equals the Euclidean norm.

4. SIMPLE MOTION

Let S denote a convex compact subset of R^k, $k > 1$, whose interior contains zero. Consider the ν pursuers

$$\dot{x}_i = u_i,\quad x_i \in R^k,\quad k>1,\quad \mu_S(u_i)\leq 1 \tag{4.1}$$

and the evader

$$\dot{y}_i = v,\quad y_i \in R^k,\quad \mu_S(v)\leq 1 \tag{4.2}$$

The game is assumed to be over if $d_s(x_i|y) = 0$ for some i. Let us reduce the problem to the form (1.1)–(1.3). To that end, put $z_i = x_i - y$. Then, instead of Eqs (4.1) and (4.2), we obtain the following conflict-controllable process

$$\dot{z}_i = u_i - v,\quad z_i \in R^k,\quad \mu_S(u_i)\leq 1,\quad \mu_S(v)\leq 1 \tag{4.3}$$

with terminal payoff function

$$\sigma(z) = \min_{1\leq i\leq\nu} d_S(z_i|0)$$

In this case $L_i = R^k$ and π_i is the identical operator, represented by the k-dimensional identity matrix. The matrices A_i are k-dimensional zero matrices and $\Phi_i(t)$ are k-dimensional identity matrices.

We first verify Pontryagin's condition

$$W_i(t,v) = S - v, \quad W_i(t) = \{0\}$$

The condition $W_i(t) = \{0\}$ uniquely defines the selectors $\gamma_i(t) \equiv 0$ and the functions $\xi_i(t, z_i, \gamma_i(\cdot)) = z_i$. Then, provided $z_i \neq 0$, we derive from (2.1) that

$$\beta_i(t,\tau,z_i,v,0) = \sup\{\beta_i \geqslant 0: \min_{u_i \in S} \max_{p_i \in S^0} [(p_i, u_i - v) + \beta_i(p_i, z_i)] \leqslant 0\},$$

$$t \geqslant \tau \geqslant 0, \quad z_i \in R^k, \quad v \in S$$

By Lemma 3.2, the resolving functions of the pursuers, when $\mu_S(z_i) > 0$, are the greatest positive roots of the equations $\mu_S(v - \beta_i z_i) = 1$ for β_i.

In particular, if S is taken to be the unit sphere centred at zero, Corollary 3.1 implies that the resolving functions may be written in the form

$$\beta_i(z_i,v) = \{(z_i,v) + [(z_i,v)^2 + \|z_i\|^2(1 - \|v\|^2)]^{1/2}\} \times \|z_i\|^{-2}$$

The time required to complete the group pursuit is

$$T(z,0) = T(z) = \min\left\{t \geqslant 0: \inf_{v(\cdot)\in\Omega_S} \max_{1\leqslant i\leqslant \nu} \int_0^\beta \beta_i(z_i, v(\tau))d\tau = 1\right\}$$

It can be shown [2] that the following inequality holds

$$\inf_{v(\cdot)\in\Omega_S} \max_{1\leqslant i\leqslant \nu} \int_0^t \beta_i(z_i, v(\tau))d\tau \geqslant \frac{t}{\nu}\delta(z), \quad \delta(z) = \min_{\mu_S(v)\leqslant 1} \max_{1\leqslant i\leqslant \nu} \beta_i(z_i, v)$$

which implies the estimate

$$T(z) \leqslant \nu/\delta(z) \tag{4.4}$$

Note that here we are considering the problem only qualitatively, that is, we wish to ensure that the group pursuit time $T(z)$ is finite, disregarding the question of minimizing the time.

Since by definition $\delta(z) \geqslant 0$ for all $z = (z_1, \ldots, z_\nu)$, $z_i \in R^k$, it follows that, to ensure that the group pursuit time $T(z)$ will be finite, we must determine those states z for which $\delta(z) > 0$.

Define

$$\rho(z) = \min_{\mu_S(v)\leqslant 1} \max_{1\leqslant i\leqslant \nu} (Z_i, v), \quad Z_i = z_i / \mu_S(z_i)$$

It is obvious that $\rho(z) > 0$ if and only if $\delta(z) > 0$. On the other hand, we have the following proposition.

Lemma 4.1. The function $\rho(z)$ is positive if and only if the zero of the space R^k lies in the interior of the convex polyhedron spanned by the vectors Z_i, that is

$$0 \in \inf \operatorname{co}\{Z_i\}_{i=1}^{\nu} \tag{4.5}$$

Proof. Suppose that (4.5) is true. This means that zero belongs to the interior of the convex polyhedron spanned by the vectors Z_i. In the terms of support functions [1], this may be expressed as follows:

$$0 < \max\nolimits_{1\leqslant i\leqslant \nu} (Z_i, v) \text{ for all } v, \mu_S(v) = 1$$

or, since the left-hand side of the inequality is independent of v, we have $\rho(z) > 0$. Reasoning in the inverse direction we get the desired result.

Theorem 4.1. Let $z^0 = (z_1^0, \ldots, z_\nu^0)$ be the initial state of process (4.3). Then, if

$$0 \in \inf \operatorname{co}\{Z_i^0\}, \quad Z_i^0 = z_i^0 / \mu_S(z_i^0) \tag{4.6}$$

the group pursuit problem is solvable at a time $T(z^0)$ for which

$$T(z^0) \leqslant \nu/\delta(z^0)$$

When that happens, if $t_* = t_*(v(\cdot))$ is the switching time, that is, a zero of the control function

$$1 - \max_{1 \leqslant i \leqslant \nu} \int_0^t \beta_i(z_i^0, v(\tau))d\tau$$

then the pursuers' controls achieving the time $T(z^0)$ in the interval $[0, t_*]$ have the form

$$u_i(\tau) = v(\tau) - \beta_i(z_i^0, v(\tau))z_i^0$$

while in the interval $(t_*, T(z^0)]$, for subscripts satisfying the equality

$$\int_0^{t_*} \beta_i(z_i^0, v(\tau))d\tau = \max_{1 \leqslant i \leqslant n} \int_0^{t_*} \beta_i(z_i^0, v(\tau))d\tau \tag{4.7}$$

the pursuers' controls are set equal to $u_i(\tau) = v(\tau)$. As to the remaining subscripts i, the pursuers' controls may be defined arbitrarily in the interval $(t_*, T(z^0)]$.

If the initial position $z^0 = (z_1^0, \ldots, z_\nu^0)$ is such that condition (4.6) fails to hold, the evader can avoid capture throughout the infinite half-interval.

Proof. Suppose that the initial position $z^0 = (z_1^0, \ldots, z_\nu^0)$ is such that (4.6) holds. Then, by Lemma 4.1, $\rho(z^0) > 0$ and so $\delta(z^0) > 0$. The fact that the time $T(z^0)$ is finite now follows from estimate (4.4). The laws by which the pursuers choose their controls before and after switching follow from the proof of Theorem 2.1. We note that in order to determine the switching time t_* at each time t, one needs information about the prehistory of the evader's control $v_t(\cdot)$. At time $T(z^0)$ each of the pursuers with subscript satisfying (4.7) will capture the evader.

Suppose that the initial position $z^0 = (z_1^0, \ldots, z_\nu^0)$ is such that (4.6) does not hold. Then $\rho(z^0) \leqslant 0$ and

$$\{\mu_S(v) = 1: \max_{1 \leqslant i \leqslant \nu} (Z_i^0, v) \leqslant 0\} \neq 0$$

Choose an element v^0 of this set as the evader's control. Then $\beta_i(z^0, v^0) = 0$. Consequently, taking (4.4) into consideration, we see that for all $\beta_i > 0$

$$\min_{u_i \in S} \max_{p_i \in S^0} [(p_i, u_i - v^0) + \beta_i(p_i, z_i^0)] > 0$$

Multiply both sides of the inequality by $t > 0$ and put $\beta_i = 1/t$. then, using the Minimax theorem [1, 10], we obtain

$$\max_{p_i \in S^0} [(p_i, z_i^0) + t \min_{u_i \in S} (p_i, u_i) - t(p_i, v^0)] > 0$$

for all $t > 0$. Since

$$t \min_{u-i \in S} (p_i, u_i) = \int_0^t \min_{u_i \in S} (p_i, u_i)d\tau = \inf_{u_i(\cdot) \in \Omega_S} \int_0^t (p_i, u_i(\cdot))d\tau$$

it follows from Cauchy's formula that for all $t > 0$

$$\inf_{u_i(\cdot) \in \Omega_S} d_S(z_i(t) | 0) > 0$$

Thus, if the evader employs the above vector v^0 as his control over a semi-infinite interval, whatever control law is chosen by the pursuer group, one has $\sigma(z(t)) > 0$, meaning that the evader will avoid capture.

Corollary 4.1. Suppose that in the simple group pursuit problem (4.3) the terminal payoff function has the form

$$\sigma(z) = \min_{1 \le i \le \nu} d_S(z_i | \varepsilon_i S), \quad \varepsilon_i > 0$$

Then, if the initial position $z^0 = (z_1^0, \dots, z_\nu^0)$ is such that $0 \in \operatorname{intco} \{z_i^0 + \varepsilon_i S\}_{i=1}^{\nu}$, at least one of the pursuers will capture the evader within a finite time. If $0 \notin \operatorname{intco} \{z^0_i + \varepsilon_i S\}_{i=1}^{\nu}$, the evader will avoid capture over a semi-infinite time interval.

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